

### 3. Wave Equation from Maxwell's Equations

#### Lossless Medium

In a source free region, Maxwell's equations are

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \quad (1)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3)$$

$$\nabla \cdot \mathbf{D} = 0, \quad (4)$$

where  $\mathbf{B} = \mu \mathbf{H}$  and  $\mathbf{D} = \epsilon \mathbf{E}$ . Taking the curl of (2), we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H}. \quad (5)$$

Substituting (1) into (5), we obtain

$$\nabla \times \nabla \times \mathbf{E} = -\mu \epsilon \frac{\partial^2}{\partial t^2} \mathbf{E}. \quad (6)$$

Making use of the vector identity that

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}, \quad (7)$$

we have

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu \epsilon \frac{\partial^2}{\partial t^2} \mathbf{E}. \quad (8)$$

Since the region is source free, and  $\nabla \cdot \mathbf{E} = 0$ , we have

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2}{\partial t^2} \mathbf{E}, \quad (9)$$

which is the vector wave equation in freespace where  $\nabla \cdot \mathbf{E} = 0$ . Similarly, we can show that

$$\nabla^2 \mathbf{H} = \mu\epsilon \frac{\partial^2}{\partial t^2} \mathbf{H} \quad (10)$$

if  $\nabla \cdot \mathbf{H} = 0$ , which is, of course, true in free space.

### Plane Wave Solutions to the Vector Wave Equations

The condition for arriving at Equation (9) is that  $\nabla \cdot \mathbf{E} = 0$ . We can have solutions of the form

$$\mathbf{E} = \hat{x}E_x(z, t), \quad (11)$$

$$\mathbf{E} = \hat{y}E_y(z, t), \quad (12)$$

but not

$$\mathbf{E} = \hat{z}E_z(z, t), \quad (13)$$

because (13) violates  $\nabla \cdot \mathbf{E} = 0$  unless  $E_z$  is independent of  $z$ . If  $\mathbf{E}$  is of the form (11), then

$$\nabla^2 \mathbf{E} = \hat{x} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_x(z, t) = \hat{x} \frac{\partial^2}{\partial z^2} E_x, \quad (14)$$

with both  $\frac{\partial^2}{\partial x^2}$  and  $\frac{\partial^2}{\partial y^2}$  equal to zero. Then (9) becomes

$$\frac{\partial^2}{\partial z^2} E_x(z, t) - \mu\epsilon \frac{\partial^2}{\partial t^2} E_x(z, t) = 0. \quad (15)$$

Similarly, if  $\mathbf{H} = \hat{y}H_y(z, t)$ , (10) becomes

$$\frac{\partial^2}{\partial z^2} H_y(z, t) - \mu\epsilon \frac{\partial^2}{\partial t^2} H_y(z, t) = 0. \quad (16)$$

Equations (15) and (16) are scalar, one dimensional wave equations of the form

$$\frac{\partial^2}{\partial z^2}y(z, t) - \frac{1}{v^2} \frac{\partial^2}{\partial t^2}y(z, t) = 0, \quad (17)$$

where  $v = 1/\sqrt{\mu\epsilon}$ . The solution to (17) is of the form  $y = f(z + at)$ . We can show that

$$\frac{\partial}{\partial z}f = f'(z + at), \quad \frac{\partial f}{\partial t} = af'(z + at), \quad (18)$$

$$\frac{\partial^2}{\partial z^2}f = f''(z + at), \quad \frac{\partial^2 f}{\partial t^2} = a^2 f''(z + at). \quad (19)$$

Substituting (19) into (17), we have

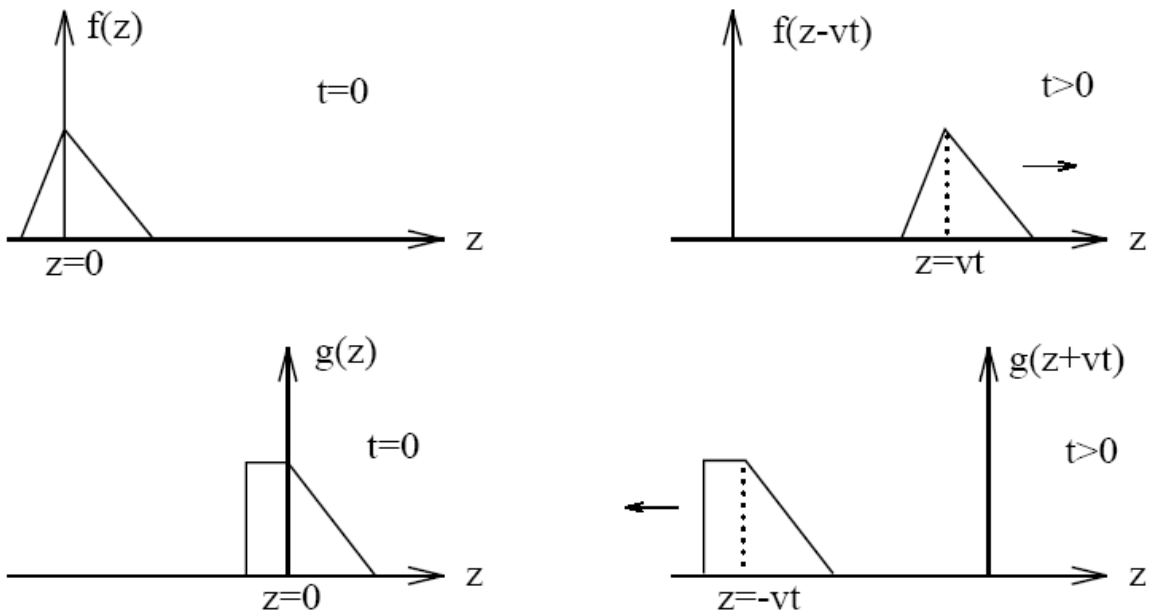
$$f''(z + at) - \frac{a^2}{v^2}f''(z + at) = 0, \quad (20)$$

which is possible only if  $a = \pm v$ . Hence, the general solution to the wave equation is

$$y = f(z - vt) + g(z + vt), \quad (21)$$

where  $f$  and  $g$  are arbitrary functions.

The solution  $f(z - vt)$  moves in the positive  $z$ -direction for increasing  $t$ .



The solution  $g(z + vt)$  moves in the negative  $z$ -direction for increasing  $t$ .

The shapes of the functions  $f$  and  $g$  are undistorted as they move along. We can observe wavelike behavior in a pond when we drop a pebble into it. Solutions to (9) and (10) that correspond to a plane wave is of the form

$$\mathbf{E} = \hat{x}f_1(z - vt), \quad \mathbf{H} = \hat{y}f_2(z - vt). \quad (22)$$

The wave is propagating in the  $z$ -direction, but the electric and magnetic fields are transverse to the direction of propagation. Such a wave is known as the **T**ransverse **E**lectro **M**agnetic wave or TEM wave.

If one substitutes (22) into Equation (2), one has

$$\nabla \times \mathbf{E} = \hat{y} \frac{\partial}{\partial z} E_x = -\mu \frac{\partial}{\partial t} \mathbf{H}, \quad (23)$$

or

$$\frac{\partial}{\partial z} f_1(z - vt) = -\mu \frac{\partial}{\partial t} f_2(z - vt), \quad (24)$$

or

$$f_1'(z - vt) = \mu v f_2'(z - vt), \quad (25)$$

or

$$f_2(z - vt) = \sqrt{\frac{\epsilon}{\mu}} f_1(z - vt). \quad (26)$$

Hence, for a plane TEM wave,

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\epsilon}} = 377 \Omega, \quad \text{for free space.} \quad (27)$$

The quantity

$$Z = \sqrt{\frac{\mu}{\epsilon}} \quad (28)$$

is also known as the intrinsic impedance of free-space.