Queuing Theory

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Definitions of Random Variable

- A function on a probability space.
- A variable whose value is determined by the outcome of a random experiment.
- "A function that assigns a numerical value to each outcome of an experiment"(Dolciani, 1988)
- The outcomes form the sample space of the Random Variable
- In statistics, a quantity that takes any of a set of values with specified probabilities.
- A random variable is a function from S, the sample space, to R, the real line; in other words, a numerical value calculated from the outcome of a random experiment.
- a function from the set of all possible outcomes of an event to some subset of the real numbers; e.g. for the event of rolling a standard die, a random variable could assign the face shown to the set {1, 2, ..., 6}
- A variable whose values are numerical events that cannot be predicted with certainty. Random variables may be continuous or discrete.
- A variable that takes on any value of a specified set with a particular probability.
- A variable which takes on values based on an underlying probability distribution.
- A quantity that might take any of a range of values (discrete or continuous) that cannot be predicted with certainty but only described probabilistically.
- A statistical variable that takes on multiple (or a continuum of) values, each with some probability that is specified by a probability distribution (or probability density function).
- A random variable is a variable in the sense that it can be used as a placeholder for a number in equations and inequalities. Its randomness is completely described by its cumulative distribution function which can be used to determine the probability it takes on particular values.
- A random variable is a real-valued function over a sample space S.

Example 1:

Suppose two dice are rolled. Let Y denote the sum of the dice.

Example 2:

Suppose two dice are rolled. Let N denote the number of rolls needed to get a snake eyes" (a sum of 2).

Definitions of Stochastic Process

Def 1: A stochastic process is a random function which varies in time for instance. Its future values cannot be precisely predicted, only with a certain amount of probability. This does not mean that the process behaves in a completely unpredictable manner, it's behaviour is governed by a random mechanism.

Def 2: A stochastic process with parameter space T is a family **{ X(t), t ∈ T }** of stochastic variables defined in the same sample space Ω . If T is an interval of real numbers, the process is said to have continuous time, if T is a sequence of integers, the process is said to have discrete time.

Def 3: A **stochastic process**, or sometimes **random process**, is the counterpart to a deterministic process in probability theory. Instead of dealing only with one possible 'reality' of how the process might evolve under time, in a stochastic process there is some indeterminacy in its future evolution described by probability distributions. This means that even if the initial condition (or starting point) is known, there are many possibilities the process might go to, but some paths are more probable and others less.

Def 4: In the simplest possible case ('discrete time'), a stochastic process amounts to a sequence of random variables known as a time series

Def 5: Another basic type of a stochastic process is a random field, whose domain is a region of space, in other words, a random function whose arguments are drawn from a range of continuously changing values.

Def 6: One approach to stochastic processes treats them as functions of one or several deterministic arguments ('inputs', in most cases regarded as 'time') whose values ('outputs') are random variables:

Stochastic process may contain more than one variable analyzed over time

Examples

Familiar examples of processes modeled as stochastic time series include

- Stock market and exchange rate fluctuations,
- Signals such as speech, audio and video,
- Medical data such as a patient's EKG, EEG, blood pressure or temperature, and
- Random movement such as Brownian motion or random walks.

Birth Death (BD) Process

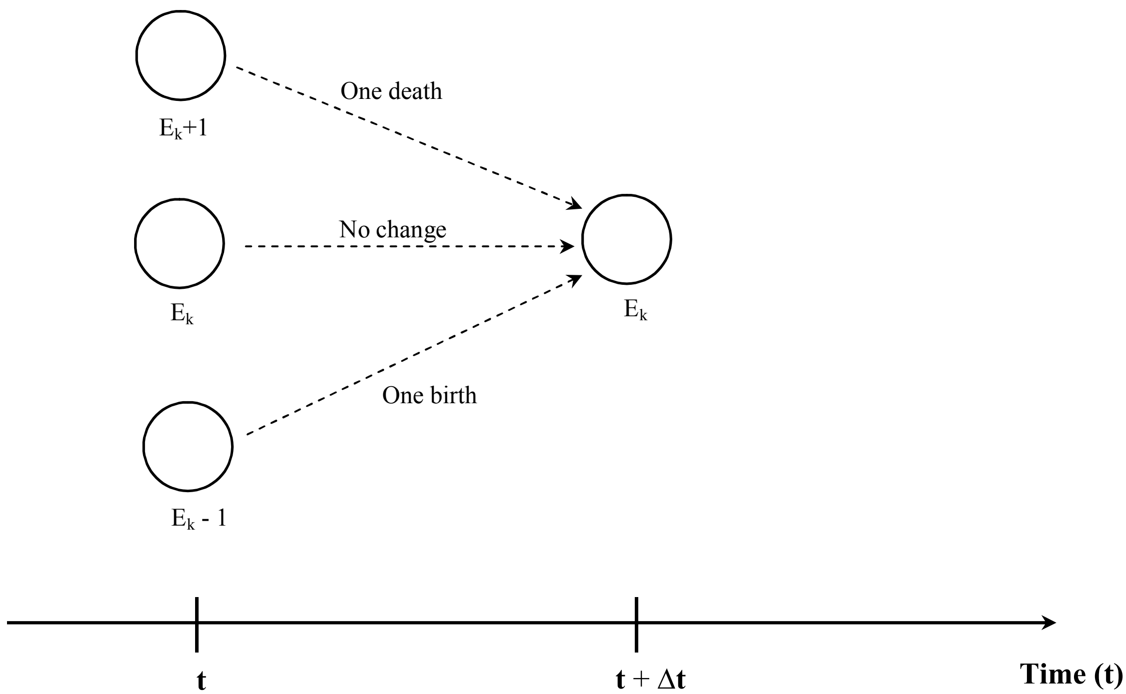
Let

λ_k = birth rate = arrival rate

μ_k = death rate = departure rate

$1/\lambda_k$ = arrival time

$1/\mu_k$ = Processing time



P[exactly one birth in (t , t + Δt)] $= \lambda_k \cdot \Delta t$ ....(1)

P[exactly one death in (t , t + Δt)] $= \mu_k \cdot \Delta t$ ....(2)

P[exactly zero birth in (t , t + Δt)] $= 1 - \lambda_k \cdot \Delta t$ ....(3)

P[exactly zero death in (t , t + Δt)] $= 1 - \mu_k \cdot \Delta t$ ....(4)

Probability of k customers in the system at time 't'

$$P_k(t) \triangleq P[\, X(t) = k \,] \quad \text{..................................(5)}$$

Where X(t) is a random variable & 'k' can be any numerical value.

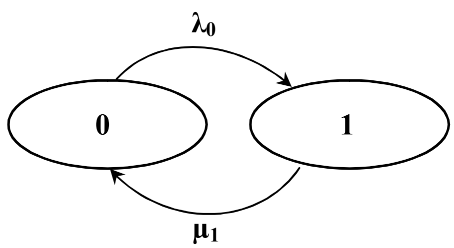
Now Probability of k customers in the system at time (t + Δt)

$$\mathbf{P_k (t + \Delta t) \ = \ P_k (t) \, . \, P_{k,k} (\Delta t) \ + \ P_{k+1} (t) \, . \, P_{k+1,k} (\Delta t) \ + \ P_{k-1} (t) \, . \, P_{k-1,k} (\Delta t)} \quad ....(6)$$

Now for Probability of zero customer in the system at time (t + Δt);

we simply put k = 0 in eqn (6)

$$\mathbf{P_0 (t + \Delta t) \ = \ P_0 (t) \, . \, P_{0,0} (\Delta t) \ + \ P_1 (t) \, . \, P_{1,0} (\Delta t) \ + 0 \text{ (-k state doesn't exist)}} \ ....(7)$$

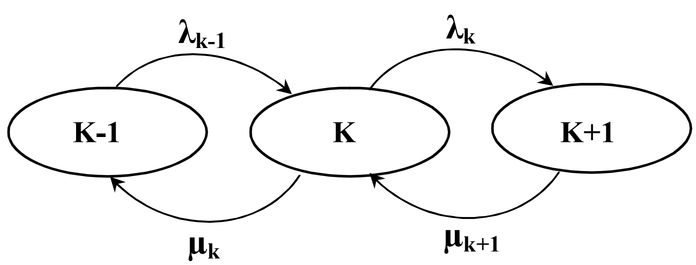


Referring back to eqn(6)

$$\mathbf{P_k (t + \Delta t) \ = \ P_k (t). \, [1 - \lambda_k \, . \, \Delta t] \, [1 - \mu_k \, . \, \Delta t] + P_{k+1} (t). \, [\mu_k . \Delta t] \ + P_{k-1} (t).[\lambda_k . \Delta t]} \ ....(8)$$

First taking the Z-Transform & then by derivation of eqn(8), we can get

$$\mathbf{- P_k (t) = \ - [\lambda_k + \mu_k] \, P_k (t) + \ \lambda_{k-1} \, . \, P_{k-1} (t) + \mu_{k+1} \, . \, P_{k+1} (t)} \ ....(9)$$



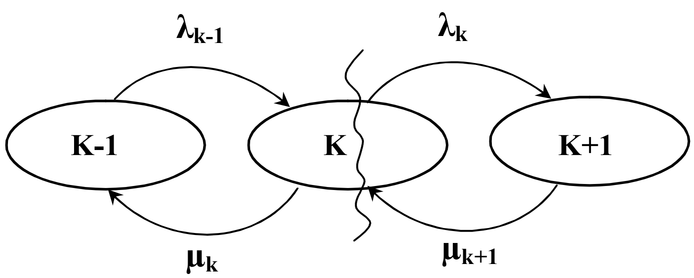
Putting k = 0 in eqn (9)

$$\mathbf{- P_0 (t) = \ - \lambda_0 \, P_0(t) + \mu_1 P_1(t)} \ ....(10)$$

Sum of probabilities of all the states can be given by

$$\sum \quad \mathbf{P_k (t) \ = \ 1} \ \ ....(11)$$

EQUATIONS OF FLOW



Making 'k' state our reference point, the equations of flow can be given by

INFLOW → $\lambda_{k-1} \cdot P_{k-1}(t) + \mu_{k+1} \cdot P_{k+1}(t)$ ....(12)

OUTFLOW → $\lambda_k \cdot P_k(t) + \mu_k \cdot P_k(t) = (\lambda_k + \mu_k).P_k(t)$ ....(13)

TIME INDEPENDENT SYSTEM

If the system is independent of time then the probability of k customers will be

$P_k(t) = \quad _{\to\infty} P_k(t)$ ....(14)

Then eqn(9) & eqn(10) becomes

$0 = -[\lambda_k + \mu_k] P_k + \lambda_{k-1} \cdot P_{k-1} + \mu_{k+1} \cdot P_{k+1}$ ....(15)

$0 = -\lambda_0 P_0 + \mu_1 P_1$ ....(16)

Finding P_1 from eqn(16)

$\lambda_0 P_0 = \mu_1 P_1$

$P_1 = \frac{\quad}{\mu}$ ....(17)

For finding P_2, putting k = 1 in eqn (15)

$$\mathbf{0 = -[\lambda_1 + \mu_1]\, P_1 + \lambda_0 \,.\, P_0 + \mu_2 \,.\, P_2}$$

$$\mathbf{0 = -\lambda_1 \,.\, P_1 - \mu_1 \,.\, P_1 + \lambda_0 \,.\, P_0 + \mu_2 \,.\, P_2}$$

$$\mathbf{\mu_2 \,.\, P_2 = \lambda_1 \,.\, P_1 + \mu_1 \,.\, P_1 - \lambda_0 \,.\, P_0}$$

$$\mathbf{P_2 = \frac{}{\mu} P_1 + \frac{\mu}{\mu} P_1 - \frac{}{\mu} P_1} \quad \text{...(18)}$$

putting value of P_1 **from eqn (17), we get**

$$\mathbf{P_2 = \frac{}{\mu}\ \frac{}{\mu}\ P_0 + \frac{\mu}{\mu}\ \frac{}{\mu}\ P_0 - \frac{}{\mu}\ \frac{}{\mu}\ P_0}$$

$$\mathbf{P_2 = P_0 \left[\frac{}{\mu\ \ \mu} + \frac{}{\mu} - \frac{}{\mu} \right]}$$

$$\mathbf{P_2 = P_0 \left[\frac{}{\mu\ \ \mu} + \frac{}{\mu} - \frac{}{\mu} \right]}$$

$$\mathbf{P_2 = \frac{}{\mu\ \ \mu}\ P_0} \quad \text{...(19)}$$

By inspecting eqn(17) & eqn(19), we can construe the probability of k customers in the system.

$$\mathbf{P_2 = \frac{\text{........}}{\mu\ \ \mu\ \ \mu\ \ \text{........}\mu}\ P_0}$$

To write in more compact form

$$\mathbf{P_k = \prod\ [\, \lambda_i / \mu_{i+1} \,] \,.\, P_0} \quad \text{...(20)}$$

Since the probability of k customers depends on P_0 (Probability of zero customers in the system) therefore we need to find it out first.

Recalling eqn (11)

$$\sum^{\infty} \quad P_k(t) = 1$$

If simply expands the summation, we get

$$P_0 + P_1 + P_2 + P_3 \ldots\ldots\ldots\ldots\ldots + P_{\infty} = 1$$

$$P_0 + \sum^{\infty} \quad P_k = 1$$

Putting values of P_k from eqn (20), we get

$$P_0 + \sum^{\infty} \quad \prod \quad [\lambda_i / \mu_{i+1}] \, . \, P_0 = 1$$

$$P_0 \; [1 + \sum^{\infty} \quad \prod \quad [\lambda_i / \mu_{i+1}]] = 1$$

$$P_0 = \frac{}{\{ \quad \sum^{\infty} \quad \prod \quad [\quad / \mu \quad]\}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21)$$

Putting value of P_0 in eqn(20), we get

$$P_k = \frac{\prod \quad [\quad / \mu \quad]}{\{ \quad \sum^{\infty} \quad \prod \quad [\quad / \mu \quad]\}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

STATE INDEPENDENT SYSTEM

So far we have covered state dependent system in which birth & death rates vary with every state but from now on we only consider state independent system in which current state contains all the necessary information as far as the future of process is concerned. Given the current state, the future of the process does not depend on its past (that is, how the process has evolved to the current state).

NOTATION FOR QUEUING MODELS

For notations of queuing models we generally follow Kendal's nomenclature which can be described as follow:

A/B/n/p/k

A refers to the ***arrival process***.
Assumption: IID inter-arrival times.
Inter-arrival time distribution:

- M= exponential (memory less)
- D= deterministic
- G= general

B refers to service times.
Assumption: IID service times.
Service time distribution:

- M= exponential (memoryless)
- D= deterministic
- G= general

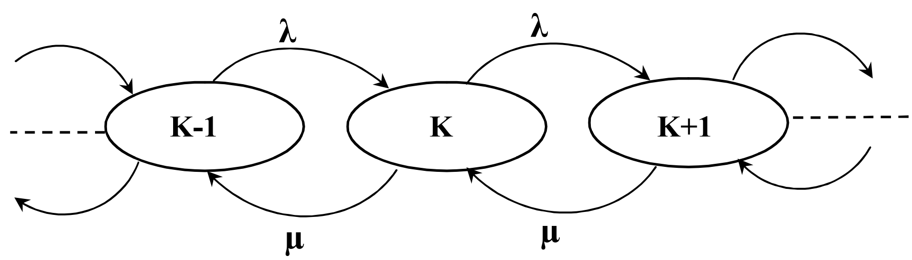
n refers to number of (parallel) servers
p refers to number of system places or number of servers + waiting places
k refers to size of customer population
• Default values (usually omitted):
$p = \infty$, $k = \infty$

• Examples:

- M/M/1
- M/D/1
- M/G/1
- G/G/1
- M/M/n
- M/M/n/n+m
- M/M/∞ (Poisson model)
- M/M/n/n (Erlang model)
- M/M/k/k/k (Binomial model)
- M/M/n/n/k (Engset model, $n < k$)

M/M/1

As described above, the first M shows, the arrival Process is Poisson or in other words the inter-arrival time follows the exponential distribution. Poisson service time is indicated by second M. There is a single processor in this model.



Here

$\lambda_k = \lambda$

$\mu_k = \mu$

We need to find out the probability of k customers in the M/M/1system

$P_k = P_0 \,.\, \prod \quad [\lambda_i / \mu_{i+1}]$

$P_k = P_0 \,.\, \frac{.\ \ .\ \ ....}{\mu.\mu.\mu.\mu..........\mu}$

$P_k = P_0 \,.\, (\lambda/\mu)^k$ ....(23)

Let $\rho = \text{—} =$ system utilization

$P_k = P_0 \,.\, (\rho)^k$ ....(24)

Here P_k depends upon P_0 therefore we also need to figure it out.

Recalling eqn (21);

$$P_0 = \frac{}{\{ \quad \Sigma^{\infty} \quad \prod \quad [\quad / \mu \quad]\}}$$

$$P_0 = \frac{}{\{ \quad \Sigma^{\infty} \quad [\ / \mu] \quad \}}$$

$$\mathbf{P_0} = \frac{}{\{ \quad \Sigma^{\infty} \quad [\;] \;\}}$$

$$\mathbf{P_0} = \frac{}{\Sigma^{\infty} \quad [\;]}$$

$$\mathbf{P_0} = \frac{}{}$$

$$\mathbf{P_0} = \mathbf{1 - \rho} \quad \text{...(25)}$$

Putting P_0 in eqn (24)

$$\mathbf{P_k} = \mathbf{(1 - \rho)} \;.\; \mathbf{(\rho)}^{k} \quad \text{...(26)}$$

Average Number of Customers in the System

$$N = \sum^{\infty} \quad k \,.\, P_k \quad \text{...(27)}$$

Putting value of P_k from eqn(26)

$$N = \sum^{\infty} \quad k \,.\, (1 - \rho) \,.\, \rho^{k}$$

$$N = \sum^{\infty} \quad k \,.\, (1 - \rho) \,.\, \rho . \rho^{k-1}$$

$$N = (1 - \rho) \,.\, \rho \; \Sigma^{\infty} \quad k \,. \rho^{k-1}$$

$$N = (1 - \rho) \,.\, \rho \; \frac{}{\rho} \; (\sum^{\infty} \quad \rho^{k})$$

$$N = (1 - \rho) \,.\, \rho \; \frac{}{\rho} \; \left(\frac{}{-\rho} \right)$$

$$N = (1 - \rho) \,.\, \rho \, [\, (1 - \rho)(0) - (-1) \,/\, (1 - \rho)^2 \,]$$

$$N = (1 - \rho) \,.\, \rho \, [\, 1 \,/\, (1 - \rho)^2 \,]$$

$$N = \frac{\rho}{-\rho} \quad \text{..(28)}$$

Average Time Delay

According to little's theorem

$$N = \lambda . T \text{ ....(29)}$$

From this eqn we can find out total delay

$$\lambda . T = \frac{\rho}{-\rho}$$

$$T = \frac{\rho}{-\rho \quad \lambda}$$

$$T = \frac{\lambda/\mu}{-\lambda/\mu \quad \lambda}$$

$$T = \frac{/\mu}{-\lambda/\mu}$$

$$T = \frac{/\mu}{-\rho} \quad \text{..(30)}$$

Probability of at least k customers in the System

$$P[\geq k \text{ in system }] = \sum^{\infty} \quad P_i$$

$$P_{k,s} = \sum^{\infty} \quad (1-\rho).\rho^k$$

$$P_{k,s} = (1-\rho) \sum^{\infty} \quad .\rho^k$$

$$P_{k,s} = (1-\rho) [\sum^{\infty} \quad .\rho^k - \sum \quad .\rho^k]$$

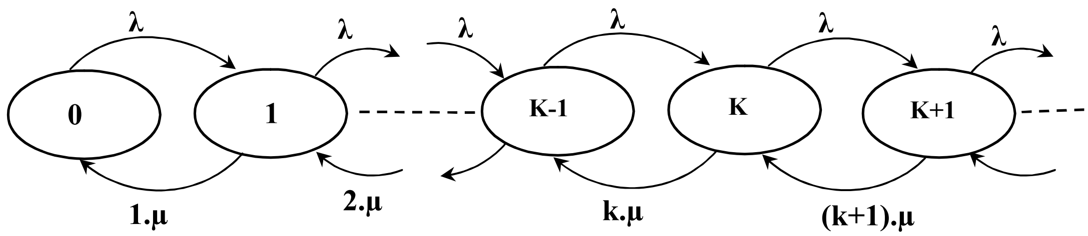
$$P_{k,s} = (1-\rho) [\frac{}{-\rho} - (1-\rho^k) / (1-\rho)]$$

$$P_{k,s} = (1-\rho) [1-1+ \rho^k / (1-\rho)]$$

$$P_{k,s} = \rho^k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

M/M/∞

In this system, we assume infinite number of processors due to which there is no queuing delay. There is no such practical system that follows M/M/∞ exactly but only practical realization can be possible. Any system may act like this model as far as the number of customers is less than number of processors.



Here

$\lambda_k = \lambda$

$\mu_k = k \, . \, \mu$

we need to find out the probability of k customers in the M/M/∞ system

$$P_k = P_0 \, . \prod \quad [\, \lambda_i / \mu_{i+1} \,]$$

$$P_k = P_0 \, . \prod \quad [\, \lambda_i / (i+1)\mu \,]$$

$$P_k = P_0 \, . \, \frac{. \quad . \quad \ldots\ldots\ldots}{\mu \quad \mu \quad \mu\ldots\ldots\ldots \quad \mu}$$

$$P_k = P_0 \, . \, (\lambda/\mu)^k / k! \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32)$$

Now finding P_0

$$P_0 = \frac{}{\{ \quad \Sigma^{\infty} \quad \prod \quad [\quad /(\quad)\mu \,] \,\}}$$

$$\mathbf{P_0 = 1 / 1 + \sum^{\infty} (\lambda/\mu)^k / k!}$$

$$\mathbf{P_0 = 1 / \sum^{\infty} (\lambda/\mu)^k / k!} \text{ ....33)}$$

$$\mathbf{P_0 = 1 / [\, 1 + (\lambda/\mu)^1 / 1! + (\lambda/\mu)^2 / 2! + (\lambda/\mu)^3 / 3! + ...]} \text{ .... (34)}$$

According to taylor's series

$$\mathbf{e^x = [\, 1 + (x)^1 / 1! + (x)^2 / 2! + (x)^3 / 3! + ...] = \sum^{\infty} (x)^k / k!}$$

If taylor's series is applied on eqn(34), we get

$$\mathbf{P_0 = 1 / e^{\lambda/\mu}}$$

$$\mathbf{P_0 = e^{-\lambda/\mu}} \text{ .... (35)}$$

Putting P_0 in eqn(32) to get

$$\mathbf{P_k = e^{-\lambda/\mu} \,.\, (\lambda/\mu)^k / k!} \text{ .... (36)}$$

The above equation shows that probability of k customers in M/M/∞ follows poisson distribution.

Total Delay (T) in M/M/∞

Total delay = Queuing delay + Processing delay

Since there is no queuing delay in this system, therefore

$$\mathbf{T = 0 + \frac{}{\mu}}$$

$$\mathbf{T = \frac{}{\mu}} \text{ .... (37)}$$

Average Number of customers (N)

From Little's theorem

$$N = \lambda . T$$

$$N = \lambda . \frac{\ }{\mu}$$

$$N = \frac{\lambda}{\mu} = \rho \quad \text{..} \ (38)$$

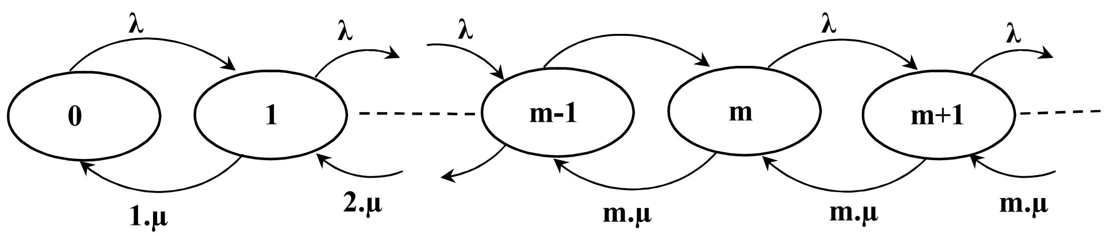
Here in this system, the average number of customers is characterized by system utilization.

M/M/m

This model is also called Erlang-C model. In this case we have finite number of processors. The customers will observe no queue until all the m processors get busy. The (m+1) customer

λ

will be held in queue. Therefore we have two cases; when number of customers 'k' is less than number of processors 'm' then we have no queuing delay and the system behaves as M/M/∞. When K is $\geq$ m the death rate becomes (m.μ)



$$\lambda_k = \lambda$$

$$\mu_k = k \cdot \mu \qquad k < m$$

$$m \cdot \mu \qquad k \geq m$$

we need to find out the probability of k customers in the M/M/m system for the following cases;

Case 1: k < m

$$P_k = P_0 \cdot \prod \quad [\lambda_i / \mu_{i+1}]$$

$$P_k = P_0 \cdot \frac{. \quad . \quad ....}{\mu \quad \mu \quad \mu.......... \quad \mu}$$

$$P_k = P_0 \cdot (\lambda/\mu)^k / k! \quad ....(39)$$

Case 2: k ≥ m

$$P_k = P_0 \cdot \prod \quad [\lambda_i / \mu_{i+1}] \cdot \prod \quad [\lambda_i / \mu_{i+1}]$$

$$P_k = P_0 \cdot \prod \quad [\lambda_i / (i+1)\mu] \cdot \prod \quad [\lambda_i / m.\mu]$$

$$P_k = P_0 \cdot [(\lambda/\mu)^m / m!] \cdot [(\lambda/\mu)^{k-m} / m^{k-m}]$$

$$P_k = P_0 \cdot [(\lambda/\mu)^{k-m+m} / (m! \cdot m^{k-m})]$$

$$P_k = P_0 \cdot [(\lambda/\mu)^k / (m! \cdot m^{k-m})]$$

$$P_k = P_0 \cdot [(\lambda/m.\mu)^k / (m! \cdot m^{-m})]$$

Here system utilization $\rho = \frac{\lambda}{\mu}$

$$P_k = P_0 \, . \, [\, \rho^k / (m! \, . \, m^{-m}) \,]$$

$$P_k = P_0 \, . \, [\, \rho^k \, . \, m^m / m! \,] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40)$$

Now finding P_0

$$P_0 = \frac{}{\{ \quad \Sigma^{\infty} \quad \Pi \quad [\quad / \mu \quad]\}} \quad \text{(general formula)}$$

$$P_0 = \frac{}{\{ \quad \Sigma \quad \Pi \quad [\quad \mu \quad] \Sigma^{\infty} \quad \Pi \quad .[\quad \mu \quad \Pi. \quad .[\quad \mu \quad]\}}$$

$$P_0 = \frac{}{\{ \Sigma \quad \Pi \quad [\quad /(\quad \mu] \,) \, \Sigma^{\infty} \quad \Pi \quad .[\quad /(\quad \mu].\Pi \quad .[\quad / \mu]. \}}$$

$$P_0 = 1 / \{ \Sigma \quad (\lambda/\mu)^k / k! \} + \{ \Sigma^{\infty} \quad [\, (\lambda/\mu)^m / m!] \, . \, [\, (\lambda/\mu)^{k-m} / m^{k-m} \,] \}$$

Directly putting value from eqn(40)

$$P_0 = 1 / \{ \Sigma \quad (\lambda/\mu)^k / k! \} + \{ \Sigma^{\infty} \quad (\rho^k \, . \, m^m / m!) \}$$

$$P_0 = 1 / \{ \Sigma \quad (m.\lambda/m.\mu)^k / k! \} + (m^m / m!) \, \{ \Sigma^{\infty} \quad \rho^k \}$$

$$P_0 = 1 / \{ \Sigma \quad (m. \, \rho)^k / k! \} + (m^m / m!) \, \{ \Sigma^{\infty} \quad \rho^k \}$$

$$P_0 = 1 / \{ \Sigma \quad (m. \, \rho)^k / k! \} + (m^m / m!) \, \{ \Sigma^{\infty} \quad \rho^k - \Sigma \quad \rho^k \}$$

$$P_0 = 1 / \{ \Sigma \quad (m. \, \rho)^k / k! \} + (m^m / m!) \, \{ \frac{}{\;} - [\, (1 - \rho^m) / (1 - \rho) \,] \}$$

$$P_0 = 1 / \{ \Sigma \quad (m. \, \rho)^k / k! \} + \{ (m^m / m!) \, [\, \rho^m / (1 - \rho) \,] \}$$

$$P_0 = 1 / \{ \Sigma \quad (m. \, \rho)^k / k! \} + \{ [(m.\rho)^m / m!] \, [(\frac{}{\;})] \} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(41)$$

Putting value of P_0 in eqn (40)

$$P_k = \{ \rho^k \, . \, m^m / m! \} / \{ \Sigma \quad (m. \, \rho)^k / k! \} + \{ [(m.\rho)^m / m!] \, [(\frac{}{\;})] \} \ldots\ldots\ldots(42)$$

Eqn (42) gives the probability of k customers in the system.

If we need to find out probability of k customers in the queue then

$$\textbf{P[k customers in queue]} = P_{kq} = \sum^{\infty} \quad P_k$$

$$P_{kq} = \sum^{\infty} \quad P_0 \, . \, [\, \rho^k \, . \, m^m / m! \,]$$

$$P_{kq} = P_0 \, (m^m/m!) \, . \sum^{\infty} \quad (\rho)^k$$

$$P_{kq} = P_0 \, (m^m/m!) \, . \, [\, \sum^{\infty} \quad \rho^k - \sum \quad \rho^k \,]$$

$$P_{kq} = P_0 \, (m^m/m!) \, . \{ \frac{}{} - [\, (1- \rho^m) / (1- \rho) \,] \, \}$$

$$P_{kq} = P_0 \, (m^m/m!) \, . \, [\, \rho^m / (1- \rho) \,]$$

$$P_{kq} = P_0 \, [(m. \, \rho)^m/m!] \, . \, [(\frac{}{})] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(43)$$

Putting value of P_0 from eqn(41) in above eqn

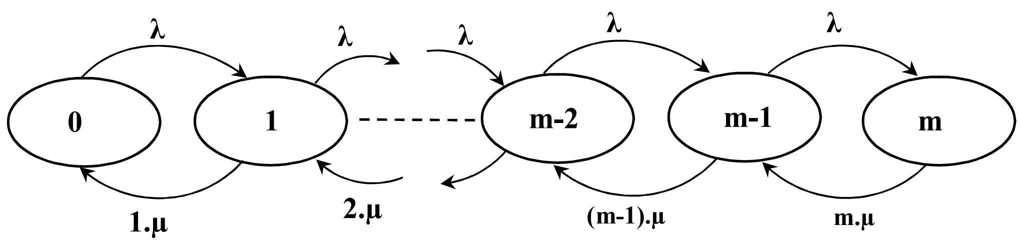
$$P_{kq} = [(m. \, \rho)^m/m!] \, . \, [(\frac{}{})] \, / \, \{ \sum \quad (m. \, \rho)^k / k! \} + \{[(m.\rho)^m / m!] \, [(\frac{}{})]\}$$

....................(44)

Eqn (44) is called **ERLANG - C** formula

M/M/m/m

This model is called Erlang-B model. The first two capital ‘M’ indicate poisson arrival process & serving time respectively. Third small ‘m’ shows finite number of processors. The last m indicates maximum number of customers allowed in the system. The system will observe no queue. Again we have two cases;



$$\lambda_k = \lambda \quad k < m$$

$$\qquad 0 \quad k \geq m$$

$$\mu_k = k \,.\, \mu$$

finding the probability of k customers in the system

$$P_k = P_0 \,.\, \prod \quad [\lambda_i / \mu_{i+1}]$$

$$P_k = P_0 \;.\; \frac{.\quad .\quad ....}{\mu \quad \mu \quad \mu.......... \quad \mu}$$

$$P_k = P_0 \;.\; (\lambda/\mu)^k / k! \text{ ....(45)}$$

Now finding P_0

$$P_0 = \frac{}{\{ \quad \Sigma^{\infty} \quad \prod \quad [\quad \mu \quad]\}} \quad \text{(general formula)}$$

$$P_0 = \frac{}{\{ \quad \Sigma^{\infty} \quad \prod \quad [\quad /(\quad \mu\,]\}}$$

$$P_0 = \frac{}{\{ \Sigma \quad \prod \quad [\quad /(\quad) \quad]\}}$$

$$P_0 = 1 / \Sigma^{\infty} \quad (\lambda/\mu)^k / k! \text{ ....(46)}$$

Putting P_0 in eqn(45)

$$P_k = (\lambda/\mu)^k / k! / \Sigma^{\infty} \quad (\lambda/\mu)^k / k! \text{ ....(47)}$$

Probability of m customers in the system can be given by

$$\mathbf{P_m} = (\lambda/\mu)^m / m! / \sum^{\infty} (\lambda/\mu)^k / k! \quad \text{.......................................}(48)$$

Eqn (48) is also called **ERLANG – B** formula